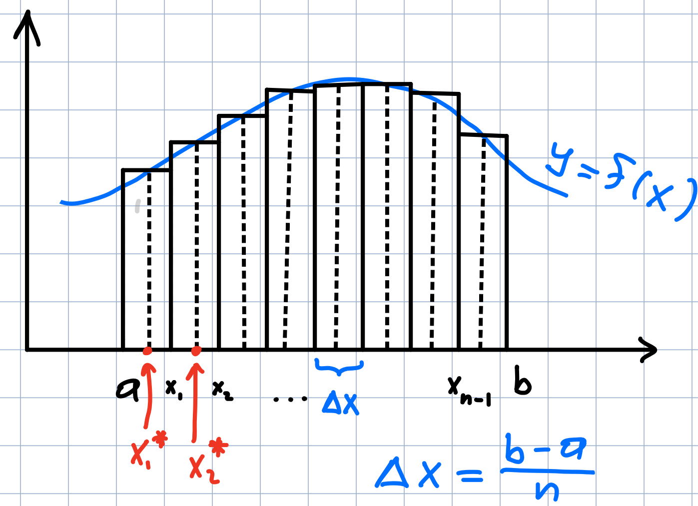
Recall:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \overbrace{f(x_i^*)\,\Delta x}^{\text{area of a rectangle}}$$

sample point in $[x_{i-1}, x_i]$



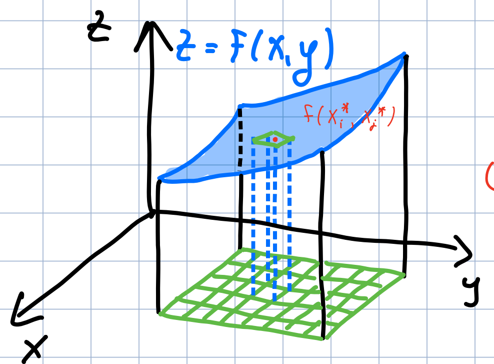
$$\Delta x = \frac{b-a}{n}$$

For $f \geq 0$

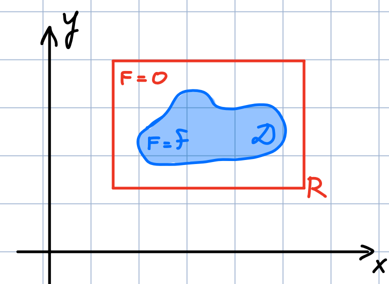
$\overset{\downarrow}{=}$ area of the region $\left\{(x,y) \,\middle|\, \begin{matrix} a \le x \le b \\ 0 \le y \le f(x) \end{matrix}\right\}$

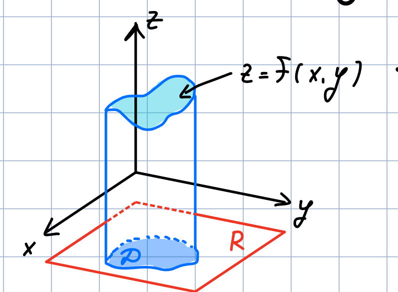
$$\iint_R f(x,y)\,dA = \lim_{\substack{m\to\infty \\ n\to\infty}} \sum_{i,j} f(x_i^*, y_j^*)\underbrace{\Delta A}_{\Delta x \cdot \Delta y} \overset{\text{for } f \geq 0}{=}$$

volume of a solid that lies above the rectangle (general region and below the surface $z=f(x,y)$)



(x_i^*, y_j^*) - sample point in $[x_{i-1}, x_i]\times[y_{j-1}, y_j]$



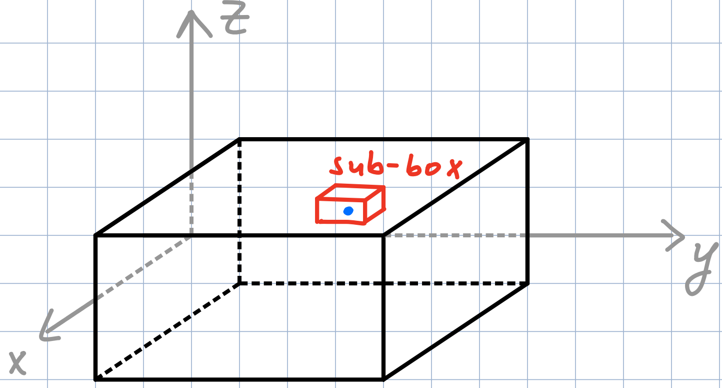


Fubini's thm: if $f(x,y)$ continuous

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

Triple integrals

$$B = \left\{ (x,y,z) \,\middle|\, \begin{matrix} a \le x \le b \\ c \le y \le d \\ r \le z \le s \end{matrix} \right\} \quad \text{rectangular box}$$



Triple integral: $f(x,y,z)$ - continuous fct.

$$\iiint_B f(x,y,z)\,dV = \lim_{l,m,n} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(\underbrace{x_{ijk}^*, y_{ijk}^*, z_{ijk}^*}_{\substack{\text{sample point in} \\ (ijk)\text{-th sub-box}}})\,\Delta V$$

Fubini's thm: if $f(x,y,z)$ continuous

$$\iiint_B f(x,y,z)\,dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz$$

Rmk: Can be written in 5 other orders, e.g.

$$\int_c^d \int_a^b \int_r^s f(x,y,z)\,dz\,dx\,dy$$

Application:

$$\iiint_E 1\,dV = \text{Volume of the solid } E$$

Ex: $B = \left\{ (x,y,z) \;\middle|\; \begin{array}{l} 0 \le x \le 1 \\ -1 \le y \le 2 \\ 0 \le z \le 3 \end{array} \right\}$ Find $\iiint_B xyz^2 \, dV$.

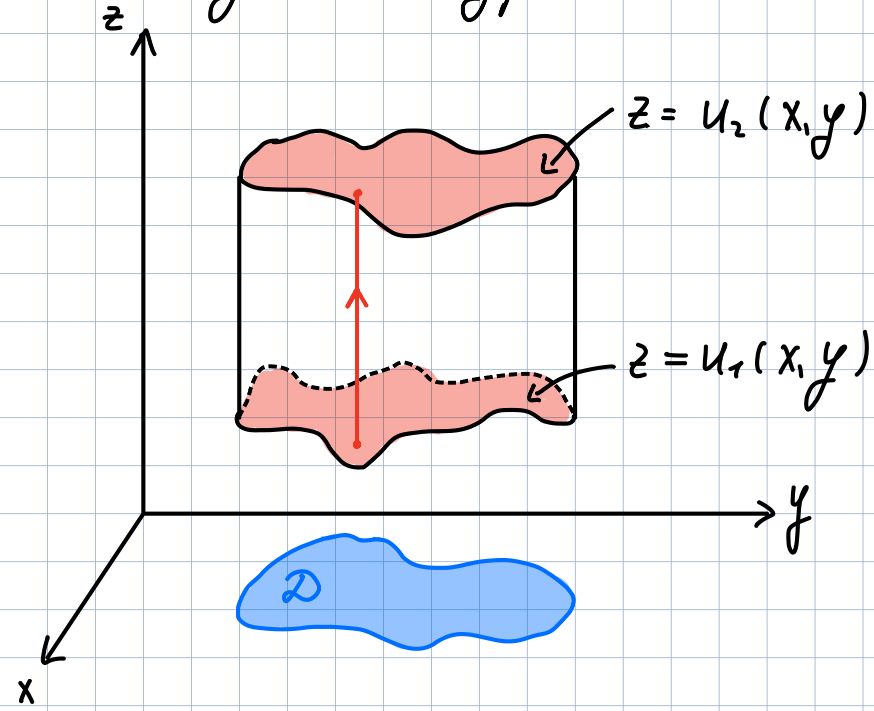
Sol:

$$\iiint_B xyz^2 \, dV = \int_0^3 \int_{-1}^2 \underbrace{\int_0^1 xyz^2 \, dx}_{\frac{1}{2}x^2yz^2 \Big|_{x=0}^{x=1} = \frac{1}{2}yz^2} \, dy \, dz$$

$$= \int_0^3 \underbrace{\int_{-1}^2 \frac{1}{2}yz^2 \, dy}_{\frac{y^2z^2}{4}\Big|_{y=-1}^{y=2} = \frac{3}{4}z^2} \, dz = \int_0^3 \frac{3}{4}z^2 \, dz = \frac{z^3}{4}\Big|_{z=0}^{z=3} = \frac{27}{4}$$

Triple integrals over general (bounded) region

Solid region of type I:



$$E = \left\{ (x,y,z) \;\middle|\; \begin{array}{l} (x,y) \in \mathcal{D} \\ u_1(x,y) \le z \le u_2(x,y) \end{array} \right\}$$

Rmk: $\mathcal{D}$ is the projection of E onto xy-plane.

$$\iiint_E f(x,y,z)\,dV = \iint_{\mathcal{D}} \left(\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right) dA$$

if $\mathcal{D}$ is of type I, then

$$E = \left\{ (x,y,z) \;\middle|\; \begin{array}{c} a \le x \le b \\ g_1(x) \le y \le g_2(x) \\ u_1(x,y) \le z \le u_2(x,y) \end{array} \right\}$$

$$= \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$

if $\mathcal{D}$ is of type II, then

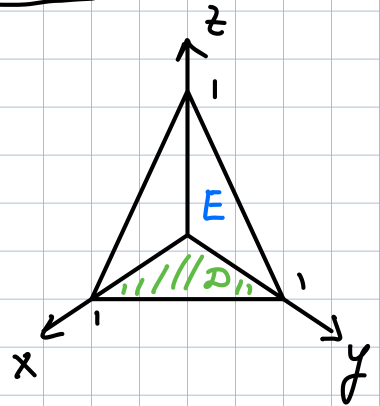
$$E = \left\{ (x,y,z) \;\middle|\; \begin{array}{c} h_1(y) \le x \le h_2(y) \\ c \le y \le d \\ u_1(x,y) \le z \le u_2(x,y) \end{array} \right\}$$

$$= \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dx\,dy$$

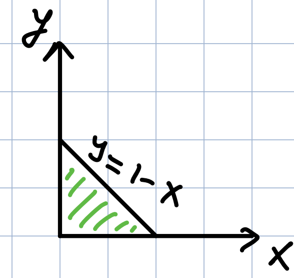
Ex: E - solid tetrahedron, bounded by $x=0,\ y=0,\ z=0$ and $x+y+z=1$.

Find $\displaystyle\iiint_E z\, dV$.

Sol:



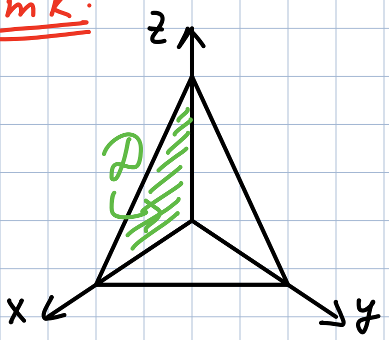
$$E = \left\{ (x,y,z) \;\middle|\; \begin{array}{l} 0 \le x \le 1 \\ 0 \le y \le 1-x \\ 0 \le z \le 1-x-y \end{array} \right\}$$



$$\iiint_E z\, dV = \iint_D \left(\int_0^{1-x-y} z\, dz \right) dA = \int_0^1 \int_0^{1-x} \underbrace{\int_0^{1-x-y} z\, dz}_{\frac{1}{2} z^2 \Big|_{z=0}^{z=1-x-y}}\, dy\, dx$$

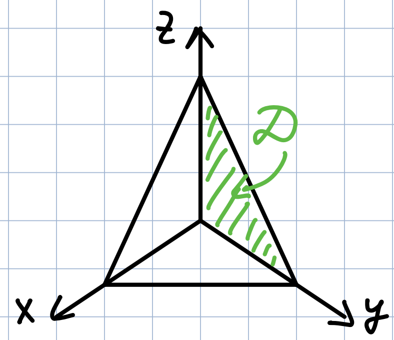
$$\int_0^1 \underbrace{\int_0^{1-x} \frac{1}{2}(1-x-y)^2\, dy}_{-\frac{1}{2}\cdot\frac{1}{3}(1-x-y)^3 \Big|_{y=0}^{y=1-x}}\, dx = \frac{1}{6} \int_0^1 (1-x)^3\, dx = -\frac{1}{6}\cdot\frac{1}{4}(1-x)^4 \Big|_{x=0}^{x=1} = \frac{1}{24}$$

Rmk:



type III

$$\iint_D \left(\int_0^{1-x-z} z\, dy \right) dA$$

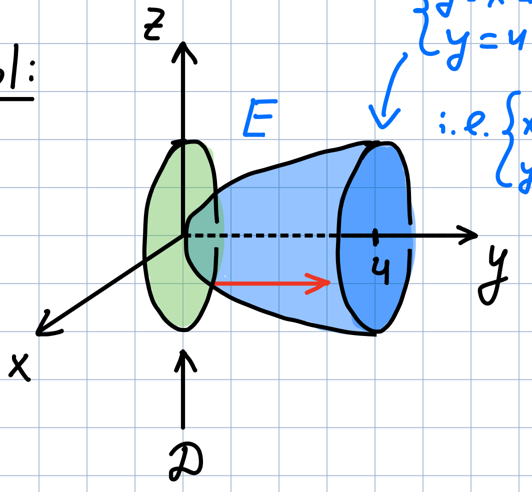


type II

$$\iint_D \left(\int_0^{1-y-z} z\, dx \right) dA$$

Ex: E bounded by $y = x^2 + z^2$ & $y = 4$. Find $\iiint_E \sqrt{x^2+z^2}\, dV$

Sol:

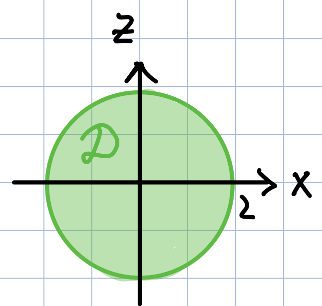


$\begin{cases} y = x^2 + z^2 \\ y = 4 \end{cases}$ i.e. $\begin{cases} x^2 + z^2 = 4 \\ y = 4 \end{cases}$

View E as "type III":

$$E = \left\{ (x,y,z) \;\middle|\; \begin{array}{l} x^2 + z^2 \leq 4 \\ x^2 + z^2 \leq y \leq 4 \end{array} \right\}$$

$$\iiint_E \sqrt{x^2+z^2}\, dV = \iint_{\mathcal{D}} \underbrace{\left(\int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\, dy \right)}_{y\sqrt{x^2+z^2}\Big|_{y=x^2+z^2}^{y=4}} dA \quad = (4 - x^2 - z^2)\sqrt{x^2+z^2}$$



$\mathcal{D}$: $0 \leq r \leq 2$
$0 \leq \theta \leq 2\pi$

$x = r\sin\theta$
$z = r\cos\theta$

$$= \iint_{\mathcal{D}} (4 - x^2 - z^2)\sqrt{x^2+z^2}\, dA = \int_0^{2\pi}\int_0^{2} (4 - r^2)\, r \cdot r\, dr\, d\theta$$

// $(4 - x^2 - z^2)\sqrt{x^2+z^2} = (4 - r^2\cos^2\theta - r^2\sin^2\theta)\sqrt{r^2\cos^2\theta + r^2\sin^2\theta}$

$= (4 - r^2)\sqrt{r^2} = r(4 - r^2)$ //

$$= \int_0^{2\pi} \underbrace{\int_0^{2} (4r^2 - r^4)\, dr}_{\frac{4}{3}r^3 - \frac{1}{5}r^5 \Big|_{r=0}^{r=2} = \frac{2}{15}\cdot 32}\, d\theta = \int_0^{2\pi} \frac{2}{15}\cdot 32\, d\theta = \frac{128}{15}\pi$$

Ex: Express $I=\int_0^1\int_0^{x^2}\int_0^{y} f(x,y,z)\,dz\,dy\,dx$

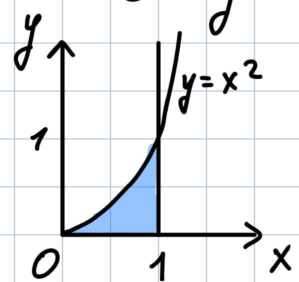
as an integral in x, then in z, then in y.

Sol: Solid $E=\left\{(x,y,z)\ \middle|\ \begin{array}{l} 0\le x\le 1\\ 0\le y\le x^2\\ 0\le z\le y\end{array}\right\}$

projections

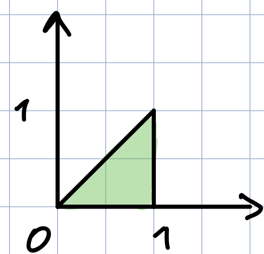
onto xy-plane

$\mathcal{D}_1:\ \begin{array}{l} 0\le x\le 1\\ 0\le y\le x^2\end{array}$



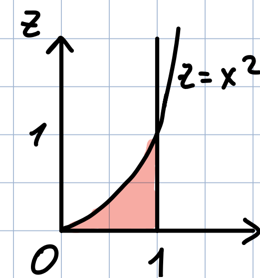
onto yz-plane

$\mathcal{D}_2:\ \begin{array}{l} 0\le y\le 1\\ 0\le z\le y\end{array}$



on xz-plane

$\mathcal{D}_3:\ \begin{array}{l} 0\le x\le 1\\ 0\le z\le x^2\end{array}$



$I=\int_0^1\int_0^{y}\int_{\sqrt{y}}^{1} f(x,y,z)\,dx\,dz\,dy \longleftrightarrow E=\left\{(x,y,z)\ \middle|\ \begin{array}{l} 0\le y\le 1\\ 0\le z\le y\\ \sqrt{y}\le x\le 1\end{array}\right\}$